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Eulerian pairs and Eulerian recurrence systems

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ABSTRACT

In this paper, we introduce the definitions of Eulerian pair and Hermite-Biehler pair. We also characterize a duality relation between Eulerian recurrences and Eulerian recurrence systems. This generalizes and unifies Hermite-Biehler decompositions of several enumerative polynomials, including up-down run polynomials for symmetric groups, alternating run polynomials for hyperoctahedral groups, flag descent polynomials for hyperoctahedral groups and flag ascent-plateau polynomials for Stirling permutations. We derive some properties of associated polynomials. In particular, we prove the alternatingly increasing property and the interlacing property of the ascent-plateau and left ascent-plateau polynomials for Stirling permutations.

1. Introduction

Let $A_n(x)$ and $B_n(x)$ be the Eulerian polynomials of types A and B, respectively. For $n \geqslant 1$, they satisfy the following recurrence relations:

$$A_n(x) = (nx + 1 - x)A_{n-1}(x) + x(1 - x)\frac{d}{dx}A_{n-1}(x),$$
$$B_n(x) = (2nx + 1 - x)B_{n-1}(x) + 2x(1 - x)\frac{d}{dx}B_{n-1}(x),$$

with $A_0(x) = B_0(x) = 1$ (see [11,12,37] for instance). In recent years, there has been much work on the generalizations of Eulerian recurrences, see [4,20,36] and references therein. For example, Salas and Villaseñor [4] classified the partial differential equations that are satisfied by the generating function

$$f(x, y) = \sum_{n,k\geqslant 0} \left|\begin{matrix} n \\ k \end{matrix}\right| x^k \frac{y^n}{n!},$$

where the numbers $\left|\begin{smallmatrix} n \\ k \end{smallmatrix}\right|$ satisfy the recurrence relation

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$$\left|\begin{matrix} n \\ k \end{matrix}\right| = (\alpha n + \beta k + \gamma)\left|\begin{matrix} n-1 \\ k-1 \end{matrix}\right| + (\alpha' n + \beta' k + \gamma')\left|\begin{matrix} n-1 \\ k \end{matrix}\right|,$$

with $\left|\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right| = 1$ and $\left|\begin{smallmatrix} n \\ k \end{smallmatrix}\right| = 0$ when $n < 0$ or $k < 0$. Very recently, Hwang, Chern and Duh [20] considered the general Eulerian recurrence:

$$\mathcal{P}_n(x) = (\alpha(x)n + \gamma(x))\mathcal{P}_{n-1}(x) + \beta(x)(1-x)\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{P}_{n-1}(x) \tag{1}$$

for $n \geqslant 1$, where $\mathcal{P}_0(x), \alpha(x), \beta(x)$ and $\gamma(x)$ are given functions of x. They studied the limiting distribution of the coefficients of $\mathcal{P}_n(x)$ for large n when the coefficients are nonnegative. In particular, Hwang, Chern and Duh [20, Section 9.3] discussed the limiting distribution of the coefficients polynomials that satisfy Eulerian recurrence systems.

We now introduce a new definition.

Definition 1. Let $\{E_n(x)\}_{n\geqslant 0}$ and $\{O_n(x)\}_{n\geqslant 0}$ be two sequences of polynomials. We say that the ordered pair of polynomials $(E_n(x), O_n(x))$ is an Eulerian pair if $\deg E_n(x) \geqslant \deg O_n(x)$ and the polynomials $E_n(x)$ and $O_n(x)$ satisfy the Eulerian recurrence system:

$$\begin{cases} E_{n+1}(x) = p_n(x)E_n(x) + q_n(x)\frac{\mathrm{d}}{\mathrm{d}x}E_n(x) + r_n(x)O_n(x), \\ O_{n+1}(x) = u_n(x)O_n(x) + v_n(x)\frac{\mathrm{d}}{\mathrm{d}x}O_n(x) + w_n(x)E_n(x), \end{cases} \tag{2}$$

where $E_0(x), O_0(x), p_n(x), q_n(x), r_n(x), u_n(x), v_n(x), w_n(x)$ are given polynomials.

Following [17], we say that a polynomial $p(x) \in \mathbb{R}[x]$ is *standard* if its leading coefficient is positive. Suppose that $p(x), q(x) \in \mathbb{R}[x]$ both have only real zeros, that those of $p(x)$ are $\xi_1 \leqslant \cdots \leqslant \xi_n$, and that those of $q(x)$ are $\theta_1 \leqslant \cdots \leqslant \theta_m$. We say that $p(x)$ *interlaces* $q(x)$ if $\deg q(x) = 1 + \deg p(x)$ and the zeros of $p(x)$ and $q(x)$ satisfy

$$\theta_1 \leqslant \xi_1 \leqslant \theta_2 \leqslant \cdots \leqslant \xi_n \leqslant \theta_{n+1}.$$

We say that $p(x)$ *alternates left of* $q(x)$ if $\deg p(x) = \deg q(x)$ and the zeros of them satisfy

$$\xi_1 \leqslant \theta_1 \leqslant \xi_2 \leqslant \cdots \leqslant \xi_n \leqslant \theta_n.$$

We use the notation $p(x) \prec_{int} q(x)$ for "$p(x)$ interlaces $q(x)$", $p(x) \prec_{alt} q(x)$ for "$p(x)$ alternates left of $q(x)$", and $p(x) \prec q(x)$ for either "$p(x) \prec_{int} q(x)$" or "$p(x) \prec_{alt} q(x)$". For notational convenience, let $a \prec bx + c$ for any real constants a, b, c.

Let $\mathbb{C}[x]$ denote the set of all polynomials in x with complex coefficients. A polynomial $p(x) \in \mathbb{C}[x]$ is *Hurwitz stable* if every zero of $p(x)$ is in the open left half plane, and $p(x)$ is *weakly Hurwitz stable* if every zero of $p(x)$ is in the closed left half of the complex plane. This concept has been extended to multivariate polynomials, see [7–9,34]. Let $\mathbb{C}[x_1, x_2, \ldots, x_n]$ denote the set of all polynomials in $x_1, x_2, \ldots, x_n$ with complex coefficients. We say that $p(x_1, x_2, \ldots, x_n) \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ is *Hurwitz stable* (resp. *weakly Hurwitz stable*) if $p(x_1, x_2, \ldots, x_n) \neq 0$ for all $(x_1, x_2, \ldots, x_n) \in \mathbb{C}^n$ with $\mathrm{Re}\, x_i \geqslant 0$ (resp. $\mathrm{Re}\, x_i > 0$), where $\mathrm{Re}\, x_i$ denote the real part of x_i.

Let $f(x) = \sum_{i=0}^n f_i x^i \in \mathbb{R}[x]$. In this paper, we always assume that

$$f^E(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} f_{2k}x^k,\ f^O(x) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} f_{2k+1}x^k;$$

$$f^e(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} f_{2k}x^{2k},\ f^o(x) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} f_{2k+1}x^{2k+1}.$$

Then $f(x) = f^E(x^2) + xf^O(x^2)$. We call this decomposition the *Hermite-Biehler decomposition* of $f(x)$, since the Hermite-Biehler theorem [16, p. 228] gives a connection between the Hurwitz stability of $f(x)$ and the interlacing property of $f^E(x)$ and $f^O(x)$. The following version of the Hermite-Biehler theorem will be used in our discussion.

Hermite-Biehler Theorem *([17, Theorem 3]). Let $f(x) = f^E(x^2) + xf^O(x^2)$ be a standard polynomial with real coefficients. Then $f(x)$ is weakly Hurwitz stable if and only if both $f^E(x)$ and $f^O(x)$ are standard, have only nonpositive zeros, and $f^O(x) \prec f^E(x)$. Moreover, $f(x)$ is Hurwitz stable if and only if $f(x)$ is weakly Hurwitz stable, $f(0) \neq 0$ and $\gcd(f^E(x), f^O(x)) = 1$.*

Now we introduce another definition.

Definition 2. We say that the ordered pair of polynomials $(G(x), H(x))$ is a Hermite-Biehler pair if $G(x)$ and $H(x)$ have the following decompositions:

$$\begin{cases} G(x) = G^E(x^2) + xG^O(x^2), \\ H(x) = G^E(x) + x^\delta G^O(x), \end{cases} \tag{3}$$

where $\delta = 0$ or $\delta = 1$.

In this paper, we consider combinatorial aspects of the Eulerian pairs and Hermite-Biehler pairs. In Section 2, we provide a connection between Eulerian pairs and Hermite-Biehler decompositions. In Section 3, we consider Eulerian pairs and Hermite-Biehler pairs associated with the up-down run polynomials for symmetric groups. In Section 4, we consider Eulerian pairs associated with the alternating run polynomials for hyperoctahedral groups. In Section 5, we consider Eulerian pairs and Hermite-Biehler pairs associated with the flag descent polynomials for hyperoctahedral groups. In Section 6, we first consider Eulerian pairs and Hermite-Biehler pairs associated with the flag ascent-plateau polynomials for Stirling permutations, we then show the alternatingly increasing property and interlacing property of the ascent-plateau and left ascent-plateau polynomials for Stirling permutations. The main results of this paper are Theorems 3, 7, 11, 14, 17, 19.

2. Relationship between Eulerian pairs and Hermite-Biehler decompositions

As an extension of (1), we define a sequence of polynomials $\{F_n(x)\}_{n\geqslant 0}$ by using the following general Eulerian recurrence:

$$F_{n+1}(x) = \alpha_n(x)F_n(x) + \beta_n(x)\frac{\mathrm{d}}{\mathrm{d}x}F_n(x), \tag{4}$$

where $F_0(x), \alpha_n(x)$ and $\beta_n(x)$ are given polynomials. We now present a fundamental result.

Theorem 3. *Let $(E_n(x), O_n(x))$ be an Eulerian pair that satisfies the Eulerian recurrence system (2), and let $F_n(x)$ be the polynomial defined by the recurrence (4). Then the polynomial $F_n(x)$ has the Hermite-Biehler decomposition $F_n(x) = E_n(x^2) + xO_n(x^2)$ if and only if the following conditions hold:*

$$u_n(x) = p_n(x) + \frac{1}{2x}q_n(x),\ v_n(x) = q_n(x),\ w_n(x) = \frac{1}{x}r_n(x),$$
$$\alpha_n(x) = p_n(x^2) + \frac{1}{x}r_n(x^2),\ \beta_n(x) = \frac{1}{2x}q_n(x^2),\ \beta_n^e(x) = 0.$$

Proof. By using $F_n(x) = E_n(x^2) + xO_n(x^2)$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}F_n(x) = 2x\frac{\mathrm{d}}{\mathrm{d}x}E_n(x^2) + O_n(x^2) + 2x^2\frac{\mathrm{d}}{\mathrm{d}x}O_n(x^2).$$

Then it follows from (4) that

$$F_{n+1}(x) = \alpha_n(x)\left(E_n(x^2) + xO_n(x^2)\right) + \beta_n(x)\left(2x\frac{\mathrm{d}}{\mathrm{d}x}E_n(x^2) + O_n(x^2) + 2x^2\frac{\mathrm{d}}{\mathrm{d}x}O_n(x^2)\right).$$

Comparing this with the expression $F_{n+1}(x) = E_{n+1}(x^2) + xO_{n+1}(x^2)$, we obtain

$$E_{n+1}(x^2) = \alpha_n^e(x)E_n(x^2) + x\alpha_n^o(x)O_n(x^2) + \beta_n^e(x)\left(O_n(x^2) + 2x^2\frac{\mathrm{d}}{\mathrm{d}x}O_n(x^2)\right) + 2x\beta_n^o(x)\frac{\mathrm{d}}{\mathrm{d}x}E_n(x^2),$$

$$O_{n+1}(x^2) = \frac{1}{x}\alpha_n^o(x)E_n(x^2) + \alpha_n^e(x)O_n(x^2) + \frac{1}{x}\beta_n^o(x)\left(O_n(x^2) + 2x^2\frac{\mathrm{d}}{\mathrm{d}x}O_n(x^2)\right) + 2\beta_n^e(x)\frac{\mathrm{d}}{\mathrm{d}x}E_n(x^2).$$

Since (2) holds, then $\beta_n^e(x) = 0$. Hence

$$\begin{cases} E_{n+1}(x^2) = \alpha_n^e(x)E_n(x^2) + 2x\beta_n^o(x)\frac{\mathrm{d}}{\mathrm{d}x}E_n(x^2) + x\alpha_n^o(x)O_n(x^2), \\ O_{n+1}(x^2) = \left(\alpha_n^e(x) + \frac{1}{x}\beta_n^o(x)\right)O_n(x^2) + 2x\beta_n^o(x)\frac{\mathrm{d}}{\mathrm{d}x}O_n(x^2) + \frac{1}{x}\alpha_n^o(x)E_n(x^2). \end{cases} \tag{5}$$

By comparing (2) with (5), we immediately get the following relations:

$$p_n(x^2) = \alpha_n^e(x),\ q_n(x^2) = 2x\beta_n^o(x),\ r_n(x^2) = x\alpha_n^o(x),$$
$$u_n(x^2) = \alpha_n^e(x) + \frac{1}{x}\beta_n^o(x),\ v_n(x^2) = 2x\beta_n^o(x),\ w_n(x^2) = \frac{1}{x}\alpha_n^o(x),$$

which yield the desired result. Conversely, when $\beta_n^e(x) = 0$, one can get the recurrence system (2) by using the following relations:

$$x\alpha_n(x) = xp_n(x^2) + r_n(x^2),\ 2x\beta_n(x) = q_n(x^2). \tag{6}$$

This completes the proof. □

3. Up-down run polynomials for symmetric groups

Let $\mathcal{S}_n$ be the set of all permutations of $[n] = \{1, 2, \ldots, n\}$ and let $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathcal{S}_n$. An *alternating run* of π is a maximal consecutive subsequence that is increasing or decreasing. The *up-down runs* of π are the alternating runs of π endowed with a 0 in the front (see [14,28]). Let $\mathrm{udrun}(\pi)$ denote the number of up-down runs of π. The numbers of *interior peaks*, *left peaks* and *valleys* of π are respectively defined as follows:

$$\mathrm{ipk}(\pi) = \#\{i \in [2, n-1] :\ \pi(i-1) < \pi(i) > \pi(i+1)\},$$
$$\mathrm{lpk}(\pi) = \#\{i \in [n-1] : \pi(i-1) < \pi(i) > \pi(i+1),\ \pi(0) = 0\},$$
$$\mathrm{val}(\pi) = \#\{i \in [2, n-1] :\ \pi(i-1) > \pi(i) < \pi(i+1)\}.$$

It is clear that $\mathrm{udrun}(\pi) = \mathrm{lpk}(\pi) + \mathrm{val}(\pi) + 1$, see [37, Lemma 2.1].

Define

$$W_n(x) = \sum_{\pi\in\mathcal{S}_n} x^{\mathrm{ipk}(\pi)},\ \overline{W}_n(x) = \sum_{\pi\in\mathcal{S}_n} x^{\mathrm{lpk}(\pi)}.$$

Note that

$$\mathrm{lpk}(\pi) = \begin{cases} \mathrm{ipk}(\pi) + 1, & \text{if } \pi(1) > \pi(2); \\ \mathrm{ipk}(\pi), & \text{otherwise.} \end{cases}$$

Then $\deg \overline{W}_n(x) \geqslant \deg W_n(x)$. For $n \geqslant 1$, the polynomials $\overline{W}_n(x)$ and $W_n(x)$ satisfy the recurrence relations

$$\overline{W}_{n+1}(x) = (nx+1)\overline{W}_n(x) + 2x(1-x)\frac{d}{dx}\overline{W}_n(x),$$
$$W_{n+1}(x) = (nx - x + 2)W_n(x) + 2x(1-x)\frac{d}{dx}W_n(x),$$

with the initial conditions $\overline{W}_1(x) = W_1(x) = 1$ (see [24,29]). Note that $(\overline{W}_n(x), W_n(x))$ is an Eulerian pair. Setting $p_n(x) = nx + 1,\ q_n(x) = 2x(1-x)$ and $r_n(x) = 0$, we get

$$p_n(x) + \frac{1}{2x}q_n(x) = nx - x + 2.$$

Then, by using Theorem 3, we can define

$$\alpha_n(x) = p_n(x^2) + \frac{1}{x}r_n(x^2) = nx^2 + 1,\ \beta_n(x) = \frac{1}{2x}q_n(x^2) = x(1-x^2).$$

So we recover the following result.

Proposition 4 *([24, Eq. (9)]). Let $\{R_n(x)\}_{n\geqslant 1}$ be a sequence of polynomials defined by the recurrence relation*

$$R_{n+1}(x) = (nx^2+1)R_n(x) + x(1-x^2)\frac{d}{dx}R_n(x), \tag{7}$$

with $R_1(x) = 1 + x$. Then $R_n(x) = \overline{W}_n(x^2) + xW_n(x^2)$.

Let $\mathrm{RZ}(I)$ the set of real-rooted polynomials all of whose zeros lie in the real interval I. According to [24, Theorem 6], we have $R_n(x) \in \mathrm{RZ}[-1, 0]$. Combining the Hermite-Biehler theorem and Proposition 4, we obtain the following result.

Corollary 5. *Both $W_n(x)$ and $\overline{W}_n(x)$ have only nonpositive zeros, and $W_n(x) \prec \overline{W}_n(x)$.*

The *up-down run polynomials* $T_n(x)$ are defined by

$$T_n(x) = \sum_{\pi \in \mathcal{S}_n} x^{\mathrm{udrun}\,(\pi)}.$$

The polynomials $T_n(x)$ satisfy the recurrence relation

$$T_{n+1}(x) = x(nx+1)T_n(x) + x\left(1-x^2\right)\frac{\mathrm{d}}{\mathrm{d}x}T_n(x), \tag{8}$$

with the initial conditions $T_0(x) = 1$ and $T_1(x) = x$ (see [25,33]). Comparing (7) with (8), it is routine to check that

$$R_n(x) = \frac{1+x}{x}T_n(x). \tag{9}$$

Set $\mathcal{S}_n^+ = \{\pi \in \mathcal{S}_n :\ \pi(n-1) > \pi(n)\}$ and $\mathcal{S}_n^- = \{\pi \in \mathcal{S}_n :\ \pi(n-1) < \pi(n)\}$. We define

$$T_n^E(x) = \sum_{\pi \in \mathcal{S}_n^+} x^{\mathrm{lpk}\,(\pi)},\ T_n^O(x) = \sum_{\pi \in \mathcal{S}_n^-} x^{\mathrm{lpk}\,(\pi)}.$$

According to [37, Lemma 2.1], one has

$$\mathrm{lpk}\,(\pi) = \left\lfloor \frac{\mathrm{udrun}\,(\pi)}{2} \right\rfloor.$$

Therefore, the following result holds.

Proposition 6. *For any $n \geqslant 1$, we have*

$$\begin{cases} T_n(x) = T_n^E(x^2) + xT_n^O(x^2), \\ \overline{W}_n(x) = T_n^E(x) + T_n^O(x). \end{cases}$$

Thus $(T_n(x), \overline{W}_n(x))$ is a Hermite-Biehler pair.

From (8), we see that $\alpha_n(x) = nx^2 + x$, $\beta_n(x) = x(1-x^2)$ and $\beta_n^e(x) = 0$. Using (6), we obtain $x(nx^2+x) = xp_n(x^2) + r_n(x^2)$ and $2x^2(1-x^2) = q_n(x^2)$. It follows from Theorem 3 that

$$p_n(x) = nx,\ q_n(x) = 2x(1-x),\ r_n(x) = x,$$
$$u_n(x) = nx + 1 - x = (n-1)x + 1,\ v_n(x) = 2x(1-x),\ w_n(x) = 1.$$

Recall that $R_n(x) \in \mathrm{RZ}[-1,0]$ (see [24, Theorem 6]). Hence $T_n(x) \in \mathrm{RZ}[-1,0]$. Therefore, combining Theorem 3 and the Hermite-Biehler theorem, we obtain the main result of this section.

Theorem 7. *For $n \geqslant 1$, the polynomials $T_n^E(x)$ and $T_n^O(x)$ satisfy the recurrence system*

$$\begin{cases} T_{n+1}^E(x) = nxT_n^E(x) + 2x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}T_n^E(x) + xT_n^O(x), \\ T_{n+1}^O(x) = ((n-1)x+1)T_n^O(x) + 2x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}T_n^O(x) + T_n^E(x), \end{cases} \tag{10}$$

with the initial conditions $T_1^E(x) = 0$ and $T_1^O(x) = 1$. Thus the ordered pairs of polynomials $(T_n^E(x), T_n^O(x))$ are Eulerian pairs. Moreover, both $T_n^E(x)$ and $T_n^O(x)$ have only nonpositive zeros and $T_n^O(x) \prec T_n^E(x)$.

4. Alternating run polynomials for signed permutations

Let $\pm[n] = [n] \cup \{\overline{1}, \ldots, \overline{n}\}$, where $\overline{i} = -i$. Let $\mathcal{S}_n^B$ be the hyperoctahedral group of rank n. Elements of $\mathcal{S}_n^B$ are signed permutations σ of the set $\pm[n]$ such that $\sigma(-i) = -\sigma(i)$ for all i. As usual, we can ignore the negative index of the σ and just write $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$. In this section, we always assume that signed permutations are prepended by 0. That is, we identify σ with the word $\sigma(0)\sigma(1)\sigma(2)\cdots\sigma(n)$, where $\sigma(0) = 0$. The numbers of *peaks* and *valleys* of σ are respectively defined by

$$\mathrm{pk}\,(\sigma) = \#\{i \in [n-1] :\ \sigma(i-1) < \sigma(i) > \sigma(i+1)\},$$
$$\mathrm{val}\,(\sigma) = \#\{i \in [n-1] : \sigma(i-1) > \sigma(i) < \sigma(i+1)\}.$$

An *alternating run* of σ is defined as a maximal interval of consecutive elements on which the elements of σ are monotonic in the order $\overline{n} < \cdots < \overline{2} < \overline{1} < 0 < 1 < 2 < \cdots < n$, see [13,35]. Let $\mathrm{altrun}\,(\sigma)$ be the number of alternating runs of σ. For example, $\mathrm{altrun}\,(03\overline{1}24\overline{5}) = 4$. It is clear that $\mathrm{altrun}\,(\sigma) = \mathrm{pk}\,(\sigma) + \mathrm{val}\,(\sigma) + 1$.

Let $C_n^+ = \{\sigma \in \mathcal{S}_n^B : \sigma(1) > 0\}$ be the set of *up-signed permutations* in $\mathcal{S}_n^B$. The *alternating run polynomials* for up-signed permutations are defined by

$$C_n(x) = \sum_{\sigma \in C_n^+} x^{\mathrm{altrun}\,(\sigma)}.$$

It should be noted that if $\mathrm{altrun}\,(\sigma) = k$, then $\mathrm{altrun}\,(-\sigma) = k$, where $-\sigma = 0\overline{\sigma(1)}\ \overline{\sigma(2)} \cdots \overline{\sigma(n)}$. Therefore, one has

$$\sum_{\sigma \in \mathcal{S}_n^B} x^{\mathrm{altrun}\,(\sigma)} = 2C_n(x).$$

Zhao [35, Theorem 4.3.1] showed that the polynomials $C_n(x)$ satisfy the recurrence relation

$$C_{n+1}(x) = (2nx^2 + 3x - 1)C_n(x) + 2x\left(1 - x^2\right)\frac{\mathrm{d}}{\mathrm{d}x}C_n(x), \tag{11}$$

with the initial condition $C_1(x) = x$. The peak and valley polynomials for up-signed permutations are respectively defined by

$$U_n(x) = \sum_{\sigma \in C_n^+} x^{\mathrm{pk}\,(\sigma)},\ V_n(x) = \sum_{\sigma \in C_n^+} x^{\mathrm{val}\,(\sigma)}.$$

According to [13, Corollary 7], they satisfy the following recurrence system:

$$\begin{cases} U_{n+1}(x) = (2nx + 1)U_n(x) + 4x(1 - x)\frac{\mathrm{d}}{\mathrm{d}x}U_n(x) + xV_n(x), \\ V_{n+1}(x) = (2nx - 2x + 3)V_n(x) + 4x(1 - x)\frac{\mathrm{d}}{\mathrm{d}x}V_n(x) + U_n(x), \end{cases}$$

with $U_0(x) = 1$ and $V_0(x) = 0$. Note that $\deg U_n(x) \geqslant \deg V_n(x)$. Thus $(U_n(x), V_n(x))$ is an Eulerian pair. Put

$$p_n(x) = 2nx + 1,\ q_n(x) = 4x(1 - x),\ r_n(x) = x,$$
$$u_n(x) = 2nx - 2x + 3,\ v_n(x) = 4x(1 - x),\ w_n(x) = 1.$$

It follows from (6) that

$$\alpha_n(x) = p_n(x^2) + \frac{1}{x}r_n(x^2) = 2nx^2 + x + 1,\ \beta_n(x) = \frac{1}{2x}q_n(x^2) = 2x(1 - x^2).$$

Let $\{\widehat{C}_n(x)\}_{n \geqslant 0}$ be the sequence of polynomials defined by the recurrence relation

$$\widehat{C}_{n+1}(x) = (2nx^2 + x + 1)\widehat{C}_n(x) + 2x(1 - x^2)\frac{d}{dx}\widehat{C}_n(x), \tag{12}$$

with $\widehat{C}_0(x) = 1$. By comparing (11) with (12), it is routine to verify that

$$\widehat{C}_n(x) = \frac{1 + x}{x}C_n(x) \text{ for } n \geqslant 1,$$

which has been proved in [13, Theorem 8]. It follows from [35, Theorem 4.3.2] that $C_n(x) \in \mathrm{RZ}[-1, 0]$. And so $\widehat{C}_n(x) \in \mathrm{RZ}[-1, 0]$. Therefore, by using Theorem 3 and the Hermite-Biehler theorem, we recover the following result.

Proposition 8 *([13, Theorem 8, Theorem 10]). For any $n \geqslant 1$, one has*

$$\widehat{C}_n(x) = U_n(x^2) + xV_n(x^2),\ V_n(x) \prec U_n(x).$$

5. Flag descent polynomials for hyperoctahedral groups

5.1. Basic definitions

Recall that a descent of $\pi \in \mathcal{S}_n$ is an index $1 \leqslant i \leqslant n - 1$ such that $\pi(i) > \pi(i + 1)$. Let $\mathrm{des}\,(\pi)$ be the number of descents of π. The *Eulerian polynomials of type A* are defined by

$$A_n(x) = \sum_{\pi \in \mathcal{S}_n} x^{\mathrm{des}\,(\pi)}.$$

For $\sigma \in \mathcal{S}_n^B$, we define two kinds of descent numbers:

$$\mathrm{des}_A(\sigma) := \#\{i \in [n-1] : \sigma(i) > \sigma(i+1)\},$$
$$\mathrm{des}_B(\sigma) := \#\{i \in [0, n-1] : \sigma(i) > \sigma(i+1),\ \sigma(0) = 0\}.$$

The *type B Eulerian polynomials* are defined by

$$B_n(x) = \sum_{\sigma \in \mathcal{S}_n^B} x^{\mathrm{des}_B(\sigma)}.$$

Let $\mathcal{S}_n^B = B_n^+ \cup B_n^-$, where $B_n^+ = \{\sigma \in \mathcal{S}_n^B : \sigma(n) > 0\}$ and $B_n^- = \{\sigma \in \mathcal{S}_n^B :\ \sigma(n) < 0\}$. The *half Eulerian polynomials* of type B are defined by

$$B_n^+(x) = \sum_{\sigma \in B_n^+} x^{\mathrm{des}_B(\sigma)},\ B_n^-(x) = \sum_{\sigma \in B_n^-} x^{\mathrm{des}_B(\sigma)}.$$

Define

$$\widehat{B}_n(x) = \sum_{k=0}^{n-1} \binom{n}{k} B_k(x)(x-1)^{n-k-1}.$$

In [21], Hyatt found that

$$\widehat{B}_n(x) = B_n^+(x),\ x^n \widehat{B}_n(1/x) = B_n^-(x)$$

for $n \geqslant 2$, which implies that $B_n(x) = \widehat{B}_n(x) + x^n \widehat{B}_n(1/x)$. Motivated by Hyatt's work, in the following we shall explore Eulerian pairs and Hermite-Biehler pairs associated with the flag descent polynomials for hyperoctahedral group.

Set $\mathcal{S}_n^B = C_n^+ \cup C_n^-$, where $C_n^+ = \{\sigma \in \mathcal{S}_n^B :\ \sigma(1) > 0\}$ and $C_n^- = \{\sigma \in \mathcal{S}_n^B :\ \sigma(1) < 0\}$. We first establish a connection between C_n^+ and B_n^+ as well as C_n^- and B_n^-, and then we consider enumerative polynomials over C_n^+ and C_n^-.

Proposition 9. *For $n \geqslant 1$, we have*

$$\sum_{\sigma \in C_n^+} x^{\mathrm{des}_B(\sigma)} = \sum_{\sigma \in B_n^+} x^{\mathrm{des}_B(\sigma)}, \tag{13}$$
$$\sum_{\sigma \in C_n^-} x^{\mathrm{des}_B(\sigma)} = \sum_{\sigma \in B_n^-} x^{\mathrm{des}_B(\sigma)}. \tag{14}$$

Proof. Define

$${}^+B_n^+ = \{\sigma \in B_n : \sigma(1) > 0,\ \sigma(n) > 0\},$$
$${}^+B_n^- = \{\sigma \in B_n : \sigma(1) > 0,\ \sigma(n) < 0\},$$
$${}^-B_n^+ = \{\sigma \in B_n : \sigma(1) < 0,\ \sigma(n) > 0\},$$
$${}^-B_n^- = \{\sigma \in B_n : \sigma(1) < 0,\ \sigma(n) < 0\}.$$

Note that $C_n^+ = {}^+B_n^+ \cup {}^+B_n^-$ and $B_n^+ = {}^+B_n^+ \cup {}^-B_n^+$. A bijection Φ from C_n^+ to B_n^+ is given as follows:

(*i*) If $\sigma \in {}^+B_n^+$, then let $\Phi(\sigma) = \sigma$;
(*ii*) For $\sigma \in {}^+B_n^-$, let k be the smallest index of σ such that $\sigma(k) > 0$ and $\sigma(k+1) < 0$. Then we define $\Phi(\sigma) = \sigma(k+1)\cdots\sigma(n)\sigma(1)\cdots\sigma(k)$.

Note that $\mathrm{des}_B(\Phi(\sigma)) = \mathrm{des}_B(\sigma)$. Hence (13) holds. And so (14) holds. □

Following [1], the *flag descent number* of $\sigma \in \mathcal{S}_n^B$ is defined by

$$\mathrm{fdes}(\sigma) := \begin{cases} 2\mathrm{des}_A(\sigma) + 1, & \text{if } \sigma(1) < 0; \\ 2\mathrm{des}_A(\sigma), & \text{otherwise.} \end{cases}$$

Clearly, $\mathrm{fdes}(\sigma) = \mathrm{des}_A(\sigma) + \mathrm{des}_B(\sigma)$. The *flag descent polynomial* is defined by

$$S_n(x) = \sum_{\sigma \in \mathcal{S}_n^B} x^{\mathrm{fdes}(\sigma)}.$$

It follows from [1, Theorem 4.4] that

$$S_n(x) = (1+x)^n A_n(x) \text{ for } n \geqslant 0. \tag{15}$$

5.2. Eulerian pairs and Hermite-Biehler pairs associated with flag descent polynomials

Let $\operatorname{neg}(\sigma) := \#\{i \in [n] : \sigma(i) < 0\}$. Consider the *q-flag descent polynomials*

$$S_n(x,q) = \sum_{\sigma \in \mathcal{S}_n^B} x^{\operatorname{fdes}(\sigma)} q^{\operatorname{neg}(\sigma)}.$$

Then $S_n(x) = S_n(x,1)$. Define

$$S_n^E(x,q) = \sum_{\sigma \in C_n^+} x^{\operatorname{des}_A(\sigma)} q^{\operatorname{neg}(\sigma)},\ S_n^O(x,q) = \sum_{\sigma \in C_n^-} x^{\operatorname{des}_A(\sigma)} q^{\operatorname{neg}(\sigma)}.$$

It is easy to see that

$$S_n(x,q) = S_n^E(x^2,q) + xS_n^O(x^2,q).$$

In [11], Brenti introduced the following q-analogue of the type B Eulerian polynomials:

$$B_n(x,q) = \sum_{\sigma \in \mathcal{S}_n^B} x^{\operatorname{des}_B(\sigma)} q^{\operatorname{neg}(\sigma)}.$$

It is clear that $B_n(x,0) = A_n(x)$ and $B_n(x,1) = B_n(x)$. The polynomials $B_n(x,q)$ satisfy the recurrence relation

$$B_{n+1}(x,q) = ((1+q)nx + qx + 1)B_n(x,q) + (1+q)x(1-x)\frac{\partial}{\partial x}B_n(x,q),$$

with $B_0(x,q) = 1$ (see [11, Theorem 3.4]). Note that $\operatorname{des}_B(\sigma) = \operatorname{des}_A(\sigma)$ for $\sigma \in C_n^+$ and $\operatorname{des}_B(\sigma) = \operatorname{des}_A(\sigma) + 1$ for $\sigma \in C_n^-$. Hence

$$B_n(x,q) = S_n^E(x,q) + xS_n^O(x,q).$$

We can now conclude the following result

Proposition 10. *Let q be a given real number. For any $n \geqslant 1$, we have*

$$\begin{cases} S_n(x,q) = S_n^E(x^2,q) + xS_n^O(x^2,q), \\ B_n(x,q) = S_n^E(x,q) + xS_n^O(x,q). \end{cases}$$

Thus the ordered pair of polynomials $(S_n(x,q), B_n(x,q))$ is a Hermite-Biehler pair.

It follows from [1, Theorem 4.3] that the flag descent polynomials $S_n(x)$ satisfy the recurrence

$$S_{n+1}(x) = (2nx^2 + x + 1)S_n(x) + x(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}S_n(x), \tag{16}$$

with $S_0(x) = 1$. Set $\alpha_n(x) = 2nx^2 + x + 1$ and $\beta_n(x) = x(1-x^2)$. Note that $\beta_n^e(x) = 0$. It follows from (6) that

$$x(2nx^2 + x + 1) = xp_n(x^2) + r_n(x^2),\ 2x^2(1-x^2) = q_n(x^2).$$

Hence $p_n(x) = 2nx + 1,\ q_n(x) = 2x(1-x),\ r_n(x) = x$. By Theorem 3, we see that

$$u_n(x) = 2nx + 1 + 1 - x = (2n-1)x + 2,\ v_n(x) = 2x(1-x),\ w_n(x) = 1.$$

It is well known that Eulerian polynomials $A_n(x) \in \mathrm{RZ}(-\infty,0)$, see [23, p. 544] for instance. It follows from (15) that $S_n(x) \in \mathrm{RZ}(-\infty,0)$. By using Theorem 3 and the Hermite-Biehler theorem, we get the main result of this section.

Theorem 11. *For $n \geqslant 1$, let $S_n(x) = S_n^E(x^2) + xS_n^O(x^2)$. Then both $S_n^O(x)$ and $S_n^E(x)$ have only nonpositive zeros, and $S_n^O(x) \prec S_n^E(x)$. Moreover, the polynomials $S_n^E(x)$ and $S_n^O(x)$ satisfy the recurrence system*

$$\begin{cases} S_{n+1}^E(x) = (2nx+1)S_n^E(x) + 2x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}S_n^E(x) + xS_n^O(x), \\ S_{n+1}^O(x) = ((2n-1)x+2)S_n^O(x) + 2x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}S_n^O(x) + S_n^E(x), \end{cases} \tag{17}$$

with $S_1^E(x) = S_1^O(x) = 1$.

Define

$$S^E(x,z)=1+\sum_{n=1}^{\infty}S_n^E(x)\frac{z^n}{n!},\ S^O(x,z)=\sum_{n=1}^{\infty}S_n^O(x)\frac{z^n}{n!}.$$

By rewriting (17) in terms of generating functions, we get

$$\begin{cases}(1-2xz)\frac{\partial}{\partial z}S^E(x,z)=S^E(x,z)+2x(1-x)\frac{\partial}{\partial x}S^E(x,z)+xS^O(x,z),\\(1-2xz)\frac{\partial}{\partial z}S^O(x,z)=(2-x)S^O(x,z)+2x(1-x)\frac{\partial}{\partial x}S^O(x,z)+S^E(x,z).\end{cases}\tag{18}$$

It is well known that the exponential generating functions of $A_n(x)$ and $B_n(x)$ are given as follows (see [11, Theorem 3.4] for instance):

$$A(x,z):=\sum_{n=0}^{\infty}A_n(x)\frac{z^n}{n!}=\frac{x-1}{x-\mathrm{e}^{(x-1)z}}=\frac{(1-x)\mathrm{e}^{(1-x)z}}{1-x\mathrm{e}^{(1-x)z}},$$

$$B(x,z):=\sum_{n=0}^{\infty}B_n(x)\frac{z^n}{n!}=\frac{(1-x)\mathrm{e}^{(1-x)z}}{1-x\mathrm{e}^{2(1-x)z}}.$$

It follows from (15) that

$$S(x,z):=\sum_{n=0}^{\infty}S_n(x)\frac{z^n}{n!}=\frac{x-1}{x-\mathrm{e}^{(x^2-1)z}}.$$

Note that

$$S(x,z)=S^E(x^2,z)+xS^O(x^2,z),\ S(-x,z)=S^E(x^2,z)-xS^O(x^2,z).$$

Then it is easy to verify that

$$S^E(x^2,z)=\frac{1}{2}\left(S(x,z)+S(-x,z)\right)=\frac{x^2-\mathrm{e}^{(x^2-1)z}}{x^2-\mathrm{e}^{2(x^2-1)z}},$$

$$S^O(x^2,z)=\frac{1}{2x}\left(S(x,z)-S(-x,z)\right)=\frac{1-\mathrm{e}^{(x^2-1)z}}{\mathrm{e}^{2(x^2-1)z}-x^2}.$$

Therefore, we obtain

$$S^E(x,z)=\frac{x-\mathrm{e}^{(x-1)z}}{x-\mathrm{e}^{2(x-1)z}},\ S^O(x,z)=\frac{1-\mathrm{e}^{(x-1)z}}{\mathrm{e}^{2(x-1)z}-x}.\tag{19}$$

Proposition 12. *We have*

$$\frac{A(x,2z)}{A(x,z)}=S^E(x,z),\ \frac{B(x,z)}{A(x,z)}=1+xS^O(x,z).$$

Proof. Note that

$$A(x,2z)=\frac{x-1}{x-\mathrm{e}^{2(x-1)z}}=\frac{x-\mathrm{e}^{(x-1)z}}{x-\mathrm{e}^{2(x-1)z}}\frac{x-1}{x-\mathrm{e}^{(x-1)z}}=S^E(x,z)A(x,z),$$

which yields the first formula. By using (19), we get

$$1+xS^O(x,z)=\frac{\mathrm{e}^{2(x-1)z}-x\mathrm{e}^{(x-1)z}}{\mathrm{e}^{2(x-1)z}-x}=\frac{1-x\mathrm{e}^{(1-x)z}}{1-x\mathrm{e}^{2(1-x)z}}.$$

It follows that

$$B(x,z)=\frac{(1-x)\mathrm{e}^{(1-x)z}}{1-x\mathrm{e}^{2(1-x)z}}=\frac{1-x\mathrm{e}^{(1-x)z}}{1-x\mathrm{e}^{2(1-x)z}}\frac{(1-x)\mathrm{e}^{(1-x)z}}{1-x\mathrm{e}^{(1-x)z}}=\left(1+xS^O(x,z)\right)A(x,z),$$

which gives the second formula. This completes the proof. □

Corollary 13. *For $n \geqslant 0$, we have*

$$2^n A_n(x) = \sum_{k=0}^{n} \binom{n}{k} A_k(x) S_{n-k}^E(x),$$

$$B_n(x) = A_n(x) + x \sum_{k=0}^{n-1} \binom{n}{k} A_k(x) S_{n-k}^O(x).$$

In recent years, the real-rootedness of the descent polynomials on signed multipermutations has been extensively studied, see [22,30]. It would be interesting to explore Eulerian pairs and Hermite-Biehler pairs associated with the descent polynomials on signed multipermutations.

6. Flag ascent-plateau polynomials for Stirling permutations

6.1. Eulerian pairs and Hermite-Biehler pairs associated with flag ascent-plateau polynomials

Stirling permutations were introduced by Gessel and Stanley [18]. A *Stirling permutation* of order n is a permutation of the multiset $\{1, 1, 2, 2, \ldots, n, n\}$ such that for each i, $1 \leqslant i \leqslant n$, all entries between the two occurrences of i are larger than i. The reader is referred to [6,19,27,28] for some recent results on Stirling permutations.

Denote by $\mathcal{Q}_n$ the set of *Stirling permutations* of order n. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_{2n} \in \mathcal{Q}_n$. The *ascent-plateau number*, *left ascent-plateau number* and *flag ascent-plateau number* of σ are respectively defined by

$$\begin{aligned} \mathrm{ap}\,(\sigma) &= \#\{i \in [2, 2n-1] : \sigma_{i-1} < \sigma_i = \sigma_{i+1}\}, \\ \mathrm{lap}\,(\sigma) &= \#\{i \in [2n-1] : \sigma_{i-1} < \sigma_i = \sigma_{i+1},\ \sigma_0 = 0\}, \\ \mathrm{fap}\,(\sigma) &= \begin{cases} 2\mathrm{ap}\,(\sigma) + 1, & \text{if } \sigma_1 = \sigma_2; \\ 2\mathrm{ap}\,(\sigma), & \text{otherwise.} \end{cases} \end{aligned}$$

Clearly, $\mathrm{fap}\,(\sigma) = \mathrm{ap}\,(\sigma) + \mathrm{lap}\,(\sigma)$. The *flag ascent-plateau polynomials* are defined by

$$L_n(x) = \sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{fap}\,(\sigma)}.$$

They satisfy the recurrence relation

$$L_{n+1}(x) = (x + 2nx^2)L_n(x) + x(1 - x^2)\frac{\mathrm{d}}{\mathrm{d}x}L_n(x), \tag{20}$$

with the initial condition $L_0(x) = 1$ (see [28, p. 14]). The first few $L_n(x)$ are

$$L_1(x) = x,\ L_2(x) = x + x^2 + x^3,\ L_3(x) = x + 3x^2 + 7x^3 + 3x^4 + x^5.$$

Set $\mathcal{Q}_n = \mathcal{Q}_n^+ \cup \mathcal{Q}_n^-$, where $\mathcal{Q}_n^+ = \{\sigma \in \mathcal{Q}_n :\ \sigma_1 < \sigma_2\}$ and $\mathcal{Q}_n^- = \{\sigma \in \mathcal{Q}_n :\ \sigma_1 = \sigma_2\}$. Note that

$$L_n(x) = \sum_{\sigma \in \mathcal{Q}_n^+} x^{2\mathrm{ap}\,(\sigma)} + x \sum_{\sigma \in \mathcal{Q}_n^-} x^{2\mathrm{ap}\,(\sigma)} = L_n^E(x^2) + xL_n^O(x^2). \tag{21}$$

From (20), we see that $\alpha_n(x) = x + 2nx^2$, $\beta_n(x) = x(1 - x^2)$ and $\beta_n^e(x) = 0$. Using (6), we obtain $x(x + 2nx^2) = xp_n(x^2) + r_n(x^2)$ and $2x^2(1 - x^2) = q_n(x^2)$. Hence

$$\begin{aligned} &p_n(x) = 2nx,\ q_n(x) = 2x(1 - x),\ r_n(x) = x, \\ &u_n(x) = 2nx + 1 - x = (2n - 1)x + 1,\ v_n(x) = 2x(1 - x),\ w_n(x) = 1. \end{aligned}$$

By using Theorem 3, we get the first main result of this section.

Theorem 14. *For $n \geqslant 1$, the polynomials $L_n^E(x)$ and $L_n^O(x)$ satisfy the recurrence system*

$$\begin{cases} L_{n+1}^E(x) = 2nxL_n^E(x) + 2x(1 - x)\frac{\mathrm{d}}{\mathrm{d}x}L_n^E(x) + xL_n^O(x), \\ L_{n+1}^O(x) = ((2n - 1)x + 1)L_n^O(x) + 2x(1 - x)\frac{\mathrm{d}}{\mathrm{d}x}L_n^O(x) + L_n^E(x), \end{cases} \tag{22}$$

with the initial conditions $L_1^E(x) = 0$ and $L_1^O(x) = 1$. Thus $(L_n^E(x), L_n^O(x))$ is an Eulerian pair.

The *ascent-plateau polynomials* and *left ascent-plateau polynomials* are defined by

$$M_n(x)=\sum_{\sigma\in\mathcal{Q}_n}x^{\mathrm{ap}\,(\sigma)},\ N_n(x)=\sum_{\sigma\in\mathcal{Q}_n}x^{\mathrm{lap}\,(\sigma)}.$$

Below are the polynomials $M_n(x)$ and $N_n(x)$ for $n\leqslant 4$:

$$M_1(x)=1,\ M_2(x)=1+2x,\ M_3(x)=1+10x+4x^2,\ M_4(x)=1+36x+60x^2+8x^3;$$
$$N_1(x)=x,\ N_2(x)=2x+x^2,\ N_3(x)=4x+10x^2+x^3,\ N_4(x)=8x+60x^2+36x^3+x^4.$$

It is well known that (see [27, p. 2] for instance):

$$M(x,t)=\sum_{n\geqslant 0}M_n(x)\frac{z^n}{n!}=\sqrt{\frac{x-1}{x-\mathrm{e}^{2z(x-1)}}},$$
$$N(x,t)=\sum_{n\geqslant 0}N_n(x)\frac{z^n}{n!}=\sqrt{\frac{1-x}{1-x\mathrm{e}^{2z(1-x)}}}.$$

According to [26, Theorem 2, Theorem 3], we have

$$M_n(x)=x^nN_n\left(\frac{1}{x}\right),\ \deg N_n(x)=1+\deg M_n(x). \tag{23}$$

By definition, we see that

$$M_n(x)=L_n^E(x)+L_n^O(x),\ N_n(x)=L_n^E(x)+xL_n^O(x). \tag{24}$$

We can now conclude the following result.

Proposition 15. *For any $n\geqslant 1$, we have*

$$\begin{cases} L_n(x)=L_n^E(x^2)+xL_n^O(x^2),\\ N_n(x)=L_n^E(x)+xL_n^O(x).\end{cases}$$

Thus the ordered pair of polynomials $(L_n(x),N_n(x))$ is a Hermite-Biehler pair.

6.2. Alternatingly increasing property

Let $f(x)=\sum_{i=0}^n f_ix^i$ be a polynomial with real coefficients. We say that $f(x)$ is *unimodal* if there exists an index m such that $f_0\leqslant f_1\leqslant\cdots\leqslant f_m\geqslant f_{m+1}\geqslant\cdots\geqslant f_n$. Such an index m is called a *mode* of $f(x)$. We say that $f(x)$ is *symmetric* if $f_i=f_{n-i}$ for all $0\leqslant i\leqslant\lfloor n/2\rfloor$. If $f(x)$ is symmetric, then it can be expanded uniquely as

$$f(x)=\sum_{k=0}^{\lfloor n/2\rfloor}\gamma_kx^k(1+x)^{n-2k},$$

and it is said to be *γ-positive* if $\gamma_k\geqslant 0$ for all $0\leqslant k\leqslant\lfloor n/2\rfloor$ (see [15, Section 2]). It is well known that γ-positivity implies unimodality, see [2] for instance. Following [31, Definition 2.9], we say that $f(x)$ is *alternatingly increasing* if

$$f_0\leqslant f_n\leqslant f_1\leqslant f_{n-1}\leqslant\cdots\leqslant f_{\lfloor\frac{n+1}{2}\rfloor}.$$

Alternatingly increasing property is a stronger property than unimodality. Recently, there has been much work on the alternatingly increasing property of Ehrhart polynomials (see [5,10,32]).

According to [27, Proposition 1], we have

$$2^nxA_n(x)=\sum_{i=0}^n\binom{n}{i}N_i(x)N_{n-i}(x),\ B_n(x)=\sum_{i=0}^n\binom{n}{i}N_i(x)M_{n-i}(x).$$

It is well known that $A_n(x)$ and $B_n(x)$ are both unimodal (see [2] for instance). Motivated by the above convolution formulas, it is natural to explore the unimodality of $M_n(x)$ and $N_n(x)$.

We now recall a very recent result on the flag ascent-plateau polynomials.

Proposition 16 *([28, Theorem 19]). The flag ascent-plateau polynomial $L_n(x)$ is semi-γ-positive. More precisely, for $n \geqslant 1$, we have*

$$L_n(x) = \sum_{k=0}^{n} L_{n,k} x^k (1+x^2)^{n-k},$$

where the numbers $L_{n,k}$ satisfy the recurrence relation

$$L_{n+1,k} = kL_{n,k} + L_{n,k-1} + 4(n-k+2)L_{n,k-2}, \tag{25}$$

with the initial conditions $L_{0,0} = L_{1,1} = 1$, $L_{0,k} = 0$ for $k \neq 0$ and $L_{1,k} = 0$ for $k \neq 1$.

Since $\deg L_n(x) = 2n-1$, we have $\deg L_n^E(x) = \deg L_n^O(x) = n-1$. Note that $L_n^E(0) = 0$ and $L_n^O(0) = 1$. By using Proposition 16, we get that both $L_n^E(x)$ and $L_n^O(x)$ are γ-positive for any $n \geqslant 1$. More precisely, we have

$$\begin{cases} L_n^E(x) = \sum_{\sigma \in \mathcal{Q}_n^+} x^{\mathrm{ap}(\sigma)} = \sum_{k=1}^{\lfloor n/2 \rfloor} L_{n,2k} x^k (1+x)^{n-2k}, \\ L_n^O(x) = \sum_{\sigma \in \mathcal{Q}_n^-} x^{\mathrm{ap}(\sigma)} = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} L_{n,2k+1} x^k (1+x)^{n-1-2k}. \end{cases} \tag{26}$$

In conclusion, we present the second main result of this section.

Theorem 17. *For any $n \geqslant 1$, both the ascent-plateau polynomial $M_n(x)$ and the left ascent-plateau polynomial $N_n(x)$ are alternatingly increasing.*

Proof. It follows from (26) that both $L_n^E(x)$ and $L_n^O(x)$ are symmetric and unimodal. When $n = 2m+1$, assume that

$$L_{2m+1}^E(x) = \ell_1 x + \ell_2 x^2 + \cdots + \ell_{m-1} x^{m-1} + \ell_m x^m + \ell_m x^{m+1} + \ell_{m-1} x^{m+2} + \cdots \ell_2 x^{2m-1} + \ell_1 x^{2m},$$
$$L_{2m+1}^O(x) = 1 + \widetilde{\ell}_1 x + \widetilde{\ell}_2 x^2 + \cdots + \widetilde{\ell}_{m-1} x^{m-1} + \widetilde{\ell}_m x^m + \widetilde{\ell}_{m-1} x^{m+1} + \cdots + \widetilde{\ell}_1 x^{2m-1} + x^{2m}.$$

Then $M_{2m+1}(x) = L_{2m+1}^E(x) + L_{2m+1}^O(x) = \sum_{i=0}^{2m} M_{2m+1,i} x^i$, where

$$M_{2m+1,i} = \begin{cases} 1, & \text{if } i = 0; \\ \ell_i + \widetilde{\ell}_i, & \text{if } 1 \leqslant i \leqslant m; \\ \ell_{2m-i+1} + \widetilde{\ell}_{2m-i}, & \text{if } m+1 \leqslant i \leqslant 2m-1; \\ \ell_1 + 1, & \text{if } i = 2m. \end{cases}$$

It is clear that $1 \leqslant \ell_1 + 1 \leqslant \ell_1 + \widetilde{\ell}_1 \leqslant \ell_2 + \widetilde{\ell}_1 \leqslant \cdots \leqslant \ell_m + \widetilde{\ell}_m$, i.e.,

$$M_{2m+1,0} \leqslant M_{2m+1,2m} \leqslant M_{2m+1,1} \leqslant M_{2m+1,2m-1} \leqslant \cdots \leqslant M_{2m+1,m}.$$

Thus $M_{2m+1}(x)$ is alternatingly increasing. From (23), we see that

$$N_{2m+1}(x) = x^{2m+1} M_{2m+1}\left(\frac{1}{x}\right),$$

which yields that $N_{2m+1}(x)$ is also alternatingly increasing. In the same way, one can verify that both $M_{2m}(x)$ and $N_{2m}(x)$ are alternatingly increasing. This completes the proof. □

6.3. Interlacing property of the ascent-plateau and left ascent-plateau polynomials

In [34], Yang and Zhang studied the real-rootedness of Eulerian polynomials via the Hermite-Biehler Theorem. Along the same lines, we shall establish the interlacing property of $M_n(x)$ and $N_n(x)$. Let $\mathbb{C}_m[x]$ denote the set of polynomials over $\mathbb{C}$ with degree less than or equal to m. In [7–9], Borcea and Brändén obtained several characterizations of linear operators preserving weakly Hurwitz stability. The following characterization will be used in our discussion.

Theorem 18 *([9, Theorem 3.3]). Let $T : \mathbb{C}_m[x] \to \mathbb{C}[x]$ be a linear operator, where m is a nonnegative integer. Then T preserves weak Hurwitz stability if and only if either*

(i) T has range of dimension at most 1 and is of the form $T(f) = \alpha(f)P$, where α is a linear functional on $\mathbb{C}_m[x]$ and P is a weakly Hurwitz stable polynomial, or

(*ii*) *The polynomial*

$$T\left[(1+xy)^m\right]=\sum_{i=0}^{m}\binom{m}{i}T(x^i)y^i \tag{27}$$

is weakly Hurwitz stable in two variables x, y.

The polynomial in (27) is called the *algebraic symbol* of T with respect to the circular domains under consideration. Combining (21) and (24), we get that

$$xM_n(x^2)+N_n(x^2)=(1+x)L_n(x). \tag{28}$$

Let $\widehat{L}_n(x)=(1+x)L_n(x)$. By using (20), it is easy to verify that for $n\geqslant 0$, one has

$$\widehat{L}_{n+1}(x)=(2n+1)x^2\widehat{L}_n(x)+x(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}\widehat{L}_n(x), \tag{29}$$

with $\widehat{L}_0(x)=1+x$. The recurrence (29) could be restated as $\widehat{L}_{n+1}(x)=T\left(\widehat{L}_n(x)\right)$, where

$$T=(2n+1)x^2+x(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}.$$

Note that $\deg\widehat{L}_n(x)=2n$ for $n\geqslant 1$. It is easy to see that T is a linear operator acting on $\mathbb{C}_{2n}[x]$. The algebraic symbol of T is given by

$$\begin{aligned}
T\left[(1+xy)^{2n}\right]&=(2n+1)x^2(1+xy)^{2n}+x(1-x^2)\sum_{i=1}^{2n}\binom{2n}{i}ix^{i-1}y^i\\
&=(2n+1)x^2(1+xy)^{2n}+2nx(1-x^2)y(1+xy)^{2n-1}\\
&=x(1+xy)^{2n-1}\left(x(1+xy)+2n(x+y)\right)\\
&=x(1+xy)^{2n}\left(x+2n\frac{x+y}{1+xy}\right).
\end{aligned}$$

If $\operatorname{Re}x>0$ and $\operatorname{Re}y>0$, then it is clear that $1+xy\neq 0$. Hence $1+xy$ is weakly Hurwitz stable in variables x, y. Let $x=a+b\mathrm{i},\ y=c+d\mathrm{i}$, where $a,\ b,\ c,\ d\in\mathbb{R}$ and $\mathrm{i}=\sqrt{-1}$. Note that

$$\begin{aligned}
\frac{x+y}{1+xy}&=\frac{a+c+(b+d)\mathrm{i}}{1+ac-bd+(ad+bc)\mathrm{i}}\,\frac{1+ac-bd-(ad+bc)\mathrm{i}}{1+ac-bd-(ad+bc)\mathrm{i}}\\
&=\frac{a+c+a^2c+b^2c+ac^2+ad^2+(b-bc^2+d-a^2d-b^2d-bd^2)\mathrm{i}}{(1+ac-bd)^2+(ad+bc)^2}.
\end{aligned}$$

If $\operatorname{Re}x=a>0$ and $\operatorname{Re}y=c>0$, then we have

$$\operatorname{Re}\frac{x+y}{1+xy}=\frac{a+c+a^2c+b^2c+ac^2+ad^2}{(1+ac-bd)^2+(ad+bc)^2}=\frac{(a+c)(1+ac)+b^2c+ad^2}{(1+ac-bd)^2+(ad+bc)^2}>0,$$

and so we have

$$\operatorname{Re}\left(x+2n\frac{x+y}{1+xy}\right)>0.$$

Therefore, $T\left[(1+xy)^{2n}\right]$ is weakly Hurwitz stable in variables x, y. By using Theorem 18, we see that the operator T preserves weakly Hurwitz stability. By induction, we get that $\widehat{L}_n(x)$ is weakly Hurwitz stable. Therefore, combining (28) and the Hermite-Biehler theorem, we get the following result.

Theorem 19. *For any* $n\geqslant 1$, *both the ascent-plateau polynomial* $M_n(x)$ *and the left ascent-plateau polynomial* $N_n(x)$ *have only real nonpositive zeros and* $M_n(x)\prec_{int} N_n(x)$.

Since $\widehat{L}_n(x)$ is weakly Hurwitz stable, the polynomial $L_n(x)$ is also weakly Hurwitz stable. It follows from (21) that both $L_n^O(x)$ and $L_n^E(x)$ have only real nonpositive zeros and $L_n^O(x)\prec L_n^E(x)$ for any $n\geqslant 1$.

7. Concluding remark

In this paper, we consider the combinatorial aspects of Eulerian pairs and Hermite-Biehler pairs. Let $\{f_n(x)\}_{n\geqslant 0}$ be a sequence of polynomials with nonnegative coefficients. Suppose that

$$f_{n+1}(x) = \left(a_1n + a_2 + (b_1n + b_2)x + (c_1n + c_2)x^2\right) f_n(x) + dx(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x} f_n(x), \tag{30}$$

where $a_1, a_2, b_1, b_2, c_1, c_2, d \in \mathbb{R}$. Then

$$\alpha_n(x) = a_1n + a_2 + (b_1n + b_2)x + (c_1n + c_2)x^2,\ \beta_n(x) = dx(1-x^2).$$

It follows from (6) that

$$p_n(x) = a_1n + a_2 + (c_1n + c_2)x,\ q_n(x) = 2dx(1-x),\ r_n(x) = (b_1n + b_2)x.$$

By using Theorem 3, we obtain

$$u_n(x) = a_1n + a_2 + d + (c_1n + c_2 - d)x,\ v_n(x) = 2dx(1-x),\ w_n(x) = b_1n + b_2,$$

and then we can derive the recurrence system of the polynomials $f_n^E(x)$ and $f_n^O(x)$.

Besides the polynomials discussed in this paper, many other enumerative polynomials also satisfy the recurrence (30), see [4,36,37] for instance. We end this paper by giving an example. Following [3, Definition 1], a tree-like tableau is a Ferrers diagram where each cell contains either 0 or 1 point with some constraints. The symmetric tableaux are tree-like tableaux which are invariant with respect to reflection through the main diagonal of their diagram. Let $b(n,k)$ be the number of symmetric tableaux of size $2n+1$ with k diagonal cells, and let $b_n(x) = \sum_{k=1}^{n+1} b(n,k)x^k$. It follows from [3, Proposition 18] that

$$b_{n+1}(x) = (n+1)x(1+x)b_n(x) + x(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x} b_n(x),$$

with the initial condition $b_0(x) = x$. By using the recurrence system of the polynomials $b_n^E(x)$ and $b_n^O(x)$, one can easily derive that $b_n^E(1) = b_n^O(1) = 2^{n-1}n!$ for $n \geqslant 1$. We leave the details to the reader. It may be interesting to explore properties of $b_n^E(x)$ and $b_n^O(x)$.

Declaration of competing interest

We declare that we have no financial and personal relationships with other people or organizations that can inappropriately influence our work, there is no professional or other personal interest of any nature or kind in any product, service and/or company that could be construed as influencing the position presented in, or the review of, the manuscript entitled “Eulerian pairs and Eulerian recurrence systems”.

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